R-matrices for $U_q\widehat{osp}(1,2)$ for highest weight representations of $U_q osp(1,2)$ for general q and q is an odd root of unity *

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Abstract

We obtain the formula for intertwining operator (R-matrix) of quantum universal enveloping superalgebra $U_q\widehat{osp}(1,2)$ for $U_q osp(1,2)$-Verma modules. By its restriction we obtain the R-matrix for two semiperiodic (semicyclic), two spin-j and spin-j and semiperiodic representations.

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The intertwining operators of quantum universal enveloping algebras [1, 2, 3] lead to integrable 2D statistical systems. For q (parameter of quantum group) isn't a root of unity they all are unified in the Universal R-matrix for a given quantum group [1, 4, 5, 6]. For q is a root of unity its formal expression fails because of singularities, arizing in this case. But intertwiners for cyclic representations [7, 8] exists in this case also [9, 10, 11, 12, 13, 14] with a spectral parameter, lying on some algebraic curve, and correspond to Chiral Potts model [15, 16]. In [17] using the method of [13] had constructed the general formula for R-matrix of $U_q\widehat{sl}(2, \mathbf{C})$ for highest weight representations of $U_q sl(2, \mathbf{C})$ both for general q and q is a root of unity.

In this work we obtain its analogue for quantum superalgebra $U_q\widehat{osp}(1, 2)$ in a slightly different way. First we obtain the intertwiner for the tenzor product of two Verma modules and then restrict it to other highest weight representations of $U_q osp(1, 2)$. The graded Yang-Baxter equations are considered also.

Quantum superalgebra $U_q osp(1, 2)$ is generated by odd elements e, f and even element k with (anti) commutation relations [18]:

$$\begin{aligned} kek^{-1} &= q^{\frac{1}{2}} e \qquad k^{\pm 1}k^{\mp 1} = 1 \\ kfk^{-1} &= q^{-\frac{1}{2}} f \\ [e, f]_+ &= \frac{k^2 - k^{-2}}{q - q^{-1}} := \{k^2\}_q, \end{aligned} \tag{1}$$

where q is a deformation parameter.

In $U_q osp(1, 2)$ exists a Hoph algebra structure:

$$\begin{aligned} \Delta(e) &= k \otimes e + e \otimes k^{-1} \qquad \Delta(k^{\pm 1}) = k^{\pm 1} \otimes k^{\pm 1} \\ \Delta(f) &= k \otimes f + f \otimes k^{-1} \end{aligned} \tag{2}$$

Recall that the multiplication in tensor product is defined by the rule:

$$(a \otimes b)(c \otimes d) = (-1)^{p(a)p(b)} ac \otimes bd,$$

where $p(a)$ is a parity of a.

From (1) and (2) it follows:

$$\Delta(f)^n = \sum_{i=0}^{n} \left(\begin{array}{c} n \\ i \end{array} \right)_{-q} (k \otimes f)^i (f \otimes k^{-1})^{n-i}, \tag{3}$$

where we use the notations

$$\left(\begin{array}{c} n \\ m \end{array} \right)_x := \frac{(m)_x!}{(n)_x!(m-n)_x!},$$

$$(n)_x! := (1)_x(2)_x \ldots (n)_x, \qquad (n)_x := \frac{x^n - 1}{x - 1}$$

We put $k = q^h$, where $[h, e] = \frac{1}{2}e$, $[h, f] = -\frac{1}{2}f$. From (1) it follows ([18]):

$$[e, f^n] = (-f)^{n-1} \frac{\left[\frac{n}{2}\right]_+ \left[2h - \frac{n-1}{2}\right]_+}{[1]_+} \tag{4}$$

$$[f, e^n] = (-e)^{n-1} \frac{\left[\frac{n}{2}\right]_+ \left[2h + \frac{n-1}{2}\right]_+}{[1]_+},$$

where we used the notation of [18]:

$$[x]_+ := \frac{q^x - (-1)^{2x} q^{-x}}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}} \qquad (-1)^{2h} = 1$$

and we denote $[a,b] := ab - (-1)^{p(a)p(b)} ba$ here and in the following.

The center of $U_q osp(1,2)$ is generated by q-deformed Casimir element [18]

$$c = -(q^{\frac{1}{2}} + q^{-\frac{1}{2}})^2 f^2 e^2 + (qk^2 + q^{-1}k^{-2}) fe + \{q^{\frac{1}{2}} k^2\}_q^2 \tag{5}$$

For q not a root of unity the finite-dimentional irreducible representations π_j are in one-to-one correspondence with the $U_q osp(1,2)$ ones. They are characterized by nonnegative integer j and called spin-$\frac{j}{2}$ representations. The action of the e, f, h in π_j is given by [18]

$$\begin{aligned} &\pi_j(h) v_m = q^{\frac{1}{2}(j-m)} v_m \qquad \pi_j(f) v_m = (-1)^m v_{m+1} \qquad \pi_j(f) v_{2j} = 0 \\ &\pi_j(e) v_m = \frac{\left[\frac{m}{2}\right]_+ \left[j + \frac{m-1}{2}\right]_+}{[1]_+} v_{m-1} \end{aligned} \tag{6}$$

Consider now the case of q is an odd root of unity. Let N is a minimnal odd integer, satisfying the condition $q^N = 1$. It follows from (1) and (4) that then e^{2N}, f^{2N}, k^{2N} belong to the center of $U_q osp(1,2)$. In the irreducible representation the values of these central elements are multiples of identity operator. Here we rewrite the action of $U_q osp(1,2)$ on cyclic (periodic) representation, which is characterized by 3 complex numbers $\lambda = q^{\frac{1}{2}\mu}, \alpha, \beta$ [19], in the following form

$$\begin{aligned} \pi_{\lambda,\alpha,\beta}(f) v_m &= v_{m+1} \qquad \pi_{\lambda,\alpha,\beta}(f) v_{2N-1} = \alpha v_0 \\ \pi_{\lambda,\alpha,\beta}(e) v_m &= \left[(-1)^{m-1} \frac{\left[\frac{m}{2}\right]_+ \left[\frac{2\mu - m + 1}{2}\right]_+}{[1]_+} + (-1)^m \alpha\beta \right] v_{m-1} \\ &\quad (m = 1 \dots 2N-1), \\ \pi_{\lambda,\alpha,\beta}(e) v_0 &= \beta v_{2N-1} \qquad \pi_{\lambda,\alpha,\beta}(k) v_m = q^{\frac{\mu - m}{2}} v_m \end{aligned} \tag{7}$$

The central elements e^{2N}, f^{2N}, k^{2N} take the values

$$\begin{aligned} a := \pi_{\lambda,\alpha,\beta}(e^{2N}) = \beta \prod_{i=1}^{2N-1} (-1)^{i-1} \left[\frac{\left[\frac{i}{2}\right]_+ \left[\frac{2\mu - i + 1}{2}\right]_+}{[1]_+} - \alpha\beta \right] \\ b := \pi_{\lambda,\alpha,\beta}(f^{2N}) = \alpha, \qquad \pi_{\lambda,\alpha,\beta}(k^{2N}) = q^\mu \end{aligned} \tag{8}$$

If a or b (α or β) vanishes then the cyclic representation converts into semiperiodic one, which is irreducible for generic values of parameters also.

The cyclic and semiperiodic representations have no analoque for the classical $osp(1,2)$.

Now we will consider the affinization $U_q \widehat{osp}(1,2)$ of $U_q osp(1,2)$. It is generated by elements e_i, f_i, k_i, $(i = 0,1)$, which satisfy the following relations

$$\begin{aligned} &k_i^{\pm 1} k_j^{\pm 1} = k_j^{\pm 1} k_i^{\pm 1}, \qquad k_i^{\pm 1} k_j^{\mp 1} = k_j^{\mp 1} k_i^{\pm 1} \quad (i,j = 0,1) \\ &k_i e_j k_i^{-1} = q^{\frac{a_{ij}}{2}} e_j \qquad k_i f_j k_i^{-1} = q^{\frac{-a_{ij}}{2}} f_j \\ &[e_i, f_j]_+ = \delta_{ij} \{k_i^2\}_q, \end{aligned} \tag{9}$$

where $a_{ij} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ is Cartan matrix of $U_q\widehat{osp}(1,2)$, and Serre relations. Here $p(e_i) = p(f_i) = 1, \quad p(k_i) = 0$.

On $U_q\widehat{osp}(1,2)$ there is a Hopf agebra structure:

$$\begin{array}{ll} \Delta(e_i) = e_i \otimes k_i^{-1} + k_i \otimes e_i & \Delta(k_i^{\pm 1}) = k_i^{\pm 1} \otimes k_i^{\pm 1} \\ \Delta(f_i) = f_i \otimes k_i^{-1} + k_i \otimes f_i & \end{array} \tag{10}$$

For any complex x there is the homomorphism ρ_x: $U_q\widehat{osp}(1,2) \to U_q osp(1,2)$:

$$\begin{array}{lll} \rho_x(e_0) = (-1)^p x f & \rho_x(f_0) = (-1)^p x^{-1} e & \rho_x(k_0^{\pm 1}) = k^{\mp 1} \\ \rho_x(e_1) = e & \rho_x(f_1) = f & \rho_x(k_1^{\pm 1}) = k^{\pm 1} \end{array} \tag{11}$$

This homomorphism converts any representation of $U_q osp(1,2)$ to parameterized representation of the affine quantum algebra $U_q\widehat{osp}(1,2)$. Using ρ_x we can construct the parametrised representation of $U_q\widehat{osp}(1,2)$ from the representation of $U_q osp(1,2)$.

The Hopf algebra structure allows to consider the action of $U_q\widehat{osp}(1,2)$ on tensor products of representations. Let π_1 and π_2 are representations of $U_q osp(1,2)$ on V_1 and V_2 respectively. Then, as it was mentioned above, $\pi_i(x_i) := \pi_i \circ \rho_{x_i}$, $i = 1,2$ are the representations of $U_q\widehat{osp}(1,2)$. Let we suppose the equivalence of $\pi_1(x_1) \otimes \pi_2(x_2)$ and $\pi_2(x_2) \otimes \pi_1(x_1)$ as an $U_q\widehat{osp}(1,2)$-modules. This means the existence of some intertwining operator $\hat{R}_{\pi_1\pi_2}(x_1, x_2)$ from $V_1 \otimes V_2$ into $V_2 \otimes V_1$ such that [2]

$$\hat{R}_{\pi_1\pi_2}(x_1,x_2)\pi_1(x_1) \otimes \pi_2(x_2)(\Delta(g)) = \pi_2(x_2) \otimes \pi_1(x_1)(\Delta(g))\hat{R}_{\pi_1\pi_2}(x_1,x_2), \tag{12}$$

where $g \in U_q\widehat{osp}(1,2)$. We put $x_2 = 1, x_1 = x$ because $\hat{R}$ depends on $\frac{x_2}{x_1}$ only. Then the equations (12) for $g = f_1, e_0$ can be represented in the following form:

$$\begin{array}{l} \hat{R}(x)(f \otimes k^{-1} + k \otimes f) = (f \otimes k^{-1} + k \otimes f)\hat{R}(x) \\ \hat{R}(x)(x(-1)^p f \otimes k + k^{-1} \otimes (-1)^p f) = ((-1)^p f \otimes k + x k^{-1} \otimes (-1)^p f)\hat{R}(x) \end{array} \tag{13}$$

From (13) we obtain:

$$\begin{aligned} \hat{R}(x)(1 \otimes f) &= [\Delta(f_1)\hat{R}(x)x((-1)^{p+1} \otimes k^2) - \Delta(e_0)\hat{R}(x)] \\ &\quad \times (x(-1)^{p+1} k \otimes k^2 - k^{-1} \otimes (-1)^{p+1})^{p+1} \\ \\ \hat{R}(x)(f \otimes 1) &= [\Delta(f_1)\hat{R}(x)(k^{-2} \otimes (-1)^{p-1}) - \Delta(e_0)\hat{R}(x)] \\ &\quad \times (k^{-2} \otimes (-1)^{p+1} k^{-1} - x(-1)^{p+1} \otimes k))^{-1}, \end{aligned} \tag{14}$$

where for simplicity we use the notations:

$$\Delta(f_i) := \pi_2(1) \otimes \pi_1(x)\,(\Delta(f_i)) \qquad \Delta(e_i) := \pi_2(1) \otimes \pi_1(x)\,(\Delta(e_i))\,.$$

If π_1 and π_2 are the highest weight $U_q osp(1,2)$-representations with highest vectors $v_0^{(1)}$ and $v_0^{(2)}$ respectively and $V_1 \otimes V_2$ decomposes into direct sum of pairwise nonequivalent irreducible representations, then we can normalize $\hat{R}(x)$ such that $\hat{R}(x)(v_0^{(1)} \otimes v_0^{(2)}) = v_0^{(2)} \otimes v_0^{(1)}$. In this case we can use (14) to obtain the recursive formula for $\hat{R}(x)(f^{r_1}v_0^{(1)} \otimes f^{r_2}v_0^{(2)})$. In fact, we obtain from (14):

$$\begin{aligned}
&\hat{R}(x)(f^{r_1} \otimes f^{r_2})v_0^{(1)} \otimes v_0^{(2)} = \\
&= (-1)^{r_1}\left[x\lambda_{(2)}^2(-1)^{p(v_0^{(1)})+r_1+1}q^{-r_2+1}\Delta(f_1) - \Delta(e_0)\right]\hat{R}(x)(f^{r_1} \otimes f^{r_2-1}) \times \\
&\times(q^{-r_2-\frac{r_1}{2}+1}x(-1)^{p+1+r_1}k \otimes k^2 - q^{\frac{r_1}{2}}(-1)^{r_2}k^{-1} \otimes (-1)^p)^{-1}v_0^{(1)} \otimes v_0^{(2)} \\
&\hat{R}(x)(f^{r_1} \otimes f^{r_2})v_0^{(1)} \otimes v_0^{(2)} = \\
&= \left[\lambda_{(1)}^{-2}(-1)^{p(v_0^{(2)})+r_2+1}q^{r_1-1}\Delta(f_1) - \Delta(e_0)\right]\hat{R}(x)(f^{r_1-1} \otimes f^{r_2}) \times \\
&\times((-1)^{r_2}q^{r_1+\frac{r_2}{2}-1}k^{-2} \otimes (-1)^{p+1}k^{-1} - q^{-\frac{r_2}{2}}x(-1)^p \otimes k)^{-1}v_0^{(1)} \otimes v_0^{(2)}
\end{aligned} \tag{15}$$

Here $\lambda_{(i)}(i = 1, 2)$ are the values of k on highest vectors $v_0^{(i)}$.Using this we obtain by induction from (15):

$$\begin{aligned}
&\hat{R}(x)(f^{r_1} \otimes f^{r_2})v_0^{(1)} \otimes v_0^{(2)} = \\
&= \frac{(-1)^{r_1r_2}}{\prod_{i=0}^{r_1+r_2-1}\left(q^{\frac{i}{2}}(\lambda_{(1)}\lambda_{(2)})^{-1}(-1)^{p_1+p_2+i} - xq^{-\frac{i}{2}}\lambda_{(1)}\lambda_{(2)}\right)} \times \\
&\times\prod_{i_2=0}^{r_2-1}\left[x\lambda_{(2)}q^{-\frac{i_2}{2}}(-1)^{p_1+i_1+1}\Delta(f_1) - \lambda_{(2)}^{-1}q^{\frac{i_2}{2}}\Delta(e_0)\right] \times \\
&\times\prod_{i_1=0}^{r_1-1}\left[\lambda_{(1)}^{-1}q^{\frac{i_1}{2}}(-1)^{p_2}\Delta(f_1) - \lambda_{(1)}q^{-\frac{i_1}{2}}\Delta(e_0)\right](v_0^{(2)} \otimes v_0^{(1)})
\end{aligned} \tag{16}$$

Let now π_1 and π_2 are Verma representations of $U_q osp(1, 2)$ on M_{λ_1} and M_{λ_2} respectively. Then the elements $\Delta(e_0)^n\Delta(f_1)^m v_0^{(1)} \otimes v_0^{(2)}$ for all nonnegative integers n and m form the basis of $M_{\lambda_1} \otimes M_{\lambda_2}$. From this and construction procedure for $\hat{R}(x)$ above follows the commutativity between $\hat{R}(x)$ and $\Delta(e_0)$, $\Delta(f_1)$ on $M_{\lambda_1} \otimes M_{\lambda_2}$. It's evident that $\hat{R}(x)$ commutes with $\Delta(k_i)$, $(i = 0, 1)$ also. We shall prove the commutativity between $\hat{R}(x)$ and $\Delta(e_1)$, $\Delta(f_0)$.

For general q (q isn't a root of unity) and λ ($\lambda \neq q^{\frac{1}{2}j}$, $j = 0, 1, \ldots$) M_λ is irreducibe. For these values of q and λ_i the tenzor product of Verma modules decomposes by the rule:

$$M_{\lambda_1} \otimes M_{\lambda_2} = \bigoplus_{n=0}^{\infty} M_{\lambda_1\lambda_2 q^{\frac{1}{2}n}} \tag{17}$$

The highest vector w_0^n of $M_{\lambda_1\lambda_2 q^{\frac{1}{2}n}}$ has the form:

$$w_0^n = \sum_{j=0}^{n} c_j^i(\lambda_i, q)f^j v_0^{(1)} \otimes f^{n-j}v_0^{(2)}, \tag{18}$$

where the coefficients $c_j^n(\lambda_i, q)$ are some rational functions on λ_i. (We can normalize w_0^n in such a way that they would became a polynomials on λ_i and λ_i^{-1}). We assert that

$$\Delta(e_1)\hat{R}(x)w_0^n = 0 \tag{19}$$

Indeed, consider this equation for $\lambda_i = q^{\frac{1}{2}l_i}$ for values of l_i, which are large enough ($min(2l_1 + 1, 2l_2 + 1) \geq n$). For this values of λ_i and general q and also for spin-$\frac{l_i}{2}$ irreps (6) $\hat{R}(x)$ exists and unique, as it had been shown in [18]. So $\hat{R}(x)$ must map the null vector of $V_{\lambda_1}\otimes V_{\lambda_2}$ to the null vector of $V_{\lambda_2}\otimes V_{\lambda_1}$. Since l_i are large enough, the "border effects" of finite dimentionality of spin-$\frac{l_i}{2}$ irreps don't play the role in (19). So, (19) is valid for infinite values of λ_i. Using expression (16) for R-matrix and (18) we conclude that left hand side of (19) can be represented in the following form:

$$\Delta(e_1)\hat{R}(x)w_0^n=\sum_{j=0}^{n-1} b_j^i(\lambda_i,q)f^jv_0^{(1)}\otimes f^{n-j}v_0^{(2)}, \tag{20}$$

where the coefficients $b_j^n(\lambda_i,q)$ as $c_j^n(\lambda_i,q)$ above are some rational functions on λ_i, that we can make a polynomials on λ_i and λ_i^{-1} by choosing an appropriate normalization of w_0^n. As they vanish for infinite many values of λ_i, they vanish trivially. So, equation (19) is valid.

Consider now the vectors w_m^n, $w_m^n=\Delta(f_1)^m w_0^n$ for any nonnegative integers n and m. They form a basis of $M_{\lambda_1}\otimes M_{\lambda_2}$ also. Note that $\hat{R}(x)$ commutes with $\Delta(e_1)$. Indeed, using (19) and (4) and the commutativity between $\hat{R}(x)$ and $\Delta(f_1),\Delta(k_1)=\Delta(k)$ we obtain

$$\begin{aligned}\Delta(e_1)\hat{R}(x)w_m^n&=\Delta(e_1)\hat{R}(x)\Delta(f_1)^m w_0^n=\Delta(e_1)\Delta(f_1)^m\hat{R}(x)w_0^n=\\&=\Delta(f_1)^{m-1}\kappa(\Delta(k))\hat{R}(x)w_0^n+(-1)^m\Delta(f_1)^m\Delta(e_1)\hat{R}(x)w_0^n=\\&=\Delta(f_1)^{m-1}\hat{R}(x)\kappa(\Delta(k))w_0^n=\hat{R}(x)\Delta(f_1)^{m-1}\kappa(\Delta(k))w_0^n=\\&=\hat{R}(x)\Delta(e_1)\Delta(f_1)^m w_0^n=\hat{R}(x)\Delta(e_1)w_m^n\end{aligned} \tag{21}$$

where for simplisity we use the notation:

$$\kappa_n(k)=\kappa_n(q^h):=(-1)^{n-1}\frac{\left[\frac{n}{2}\right]_+\left[2h-\frac{n-1}{2}\right]_+}{[1]_+}$$

So, $[D(e_1),\hat{R}(x)]=0$, because w_m^n form the basis of $M_{\lambda_1}\otimes M_{\lambda_2}$.

The same result may be obtained for $\Delta(f_0)$. To derive it we have to consider the $U_q osp(1,2)$-decomposition of $M_{\lambda_1}\otimes M_{\lambda_2}$ with respect to $U_q osp(1,2)$, generated by $\Delta(e_0),\Delta(f_0),\Delta(k_0)$. (Note that in this case we deal with lowest weight Verma modules). The basis vectors w_m^n must be replaced then by the basis vectors $\tilde{w_m^n}=\Delta(f_0)^m\tilde{w_0^n}$, where $\Delta(f_0)\tilde{w_0^n}=0$, and

$$\Delta(k_0)\tilde{w_0^n}=(\lambda_1\lambda_2)^{-1}q^{\frac{1}{2}n}\tilde{w_0^n}.$$

The equation like (21) may be written in this case also.

So, we proved that $\hat{R}(x)$ is a $U_q\widehat{osp}(1,2)$-intertwining operator:

$$\hat{R}(x):M_{\lambda_1}\otimes M_{\lambda_2}\to M_{\lambda_2}\otimes M_{\lambda_1}.$$

The graded Yang-Baxter equations on $M_{\lambda_1}\otimes M_{\lambda_2}\otimes M_{\lambda_3}$

$$(id\otimes\hat{R}(x))(\hat{R}(xy)\otimes id)(id\otimes\hat{R}(y)=(\hat{R}(y)\otimes id)(id\otimes\hat{R}(xy))(\hat{R}(x)\otimes id) \tag{22}$$

follows from the $U_q\widehat{osp}(1,2)$-irreducibility of tenzor product, because both the left and right sades of (22) are $U_q\widehat{osp}(1,2)$-intertwiners: $M_{\lambda_1}\otimes M_{\lambda_2}\otimes M_{\lambda_3}\to M_{\lambda_1}\otimes M_{\lambda_2}\otimes M_{\lambda_3}$ and both they map the vector $v_0^{(1)}\otimes v_0^{(2)}\otimes v_0^{(3)}$ to the vector $v_0^{(3)}\otimes v_0^{(2)}\otimes v_0^{(1)}$.

The above constucted R-matrix (16), intertwining Verma modules, have the universality property in the following sence. Every highest weight $U_q osp(1,2)$-module V_λ can be obtained by factorizing of M_λ on some submodule I_λ: $V_\lambda = M_\lambda/I_\lambda$. Consider the restriction of equation (12) on

$$V_{\lambda_1}\otimes V_{\lambda_2} = M_{\lambda_1}/I^1_{\lambda_1}\otimes M_{\lambda_2}/I^2_{\lambda_2}. \tag{23}$$

If (12) preserves the factorization, then $\hat{R}(x)$ is well defined on (23). In a such way $\hat{R}_{V_1\otimes V_2}(x)$ can be constructed from $\hat{R}_{M_{\lambda_1}\otimes M_{\lambda_2}}$ by the restriction.

For instance, let us obtain an R-matrix for semiperiodic representation $\pi_{\lambda,\alpha} := \pi_{\lambda,\alpha,0}$ for $q^N = 1$, where N is a minimal odd integer, satisfying this condition. It characterized by two complex numbers λ and α (See equations (7) for $\beta = 0$). It is obtained from M_λ by the factorization on the submodule, generated by the vector $(f^{2N}-\alpha)v_0$: $\quad V_{\lambda\alpha} = M_\lambda/I^\alpha_\lambda$. From (3) follows:

$$\begin{aligned}\Delta(f_1)^{2N} &= k^{2N}\otimes f^{2N} + f^{2N}\otimes k^{-2N}\\ \Delta(f_0)^{2N} &= -k^{-2N}\otimes e^{2N} - xe^{2N}\otimes k^{2N}\end{aligned} \tag{24}$$

The nesessary condition for constincensy of (13) with the factorization procedure is the coincidence of central elements $\Delta(e_0)^{2N}$ and $\Delta(f_1)^{2N}$ in $V_{\lambda_1\alpha_1}\otimes V_{\lambda_2\alpha_2}$ and $V_{\lambda_2\alpha_2}\otimes V_{\lambda_1\alpha_1}$. Thus, from (24) follows

$$\frac{\alpha_1}{\lambda_1^{2N}-\lambda_1^{-2N}} = \frac{\alpha_2}{\lambda_2^{2N}-\lambda_2^{-2N}}, \qquad x^{2N} = 1 \tag{25}$$

But (25) is a suffitient condition also. Indeed, from (24),(13),(25) follows

$$\begin{aligned}&\hat{R}(x)\left[\lambda_1^{2N}(1\otimes(f^{2N}-\alpha_2)) + \lambda_2^{-2N}((f^{2N}-\alpha_1)\otimes 1\right] =\\ &\quad = \left[\lambda_2^{2N}(1\otimes(f^{2N}-\alpha_1)) + \lambda_1^{-2N}((f^{2N}-\alpha_2)\otimes 1\right]\hat{R}(x)\\ &\hat{R}(x)\left[\lambda_1^{-2N}(1\otimes(f^{2N}-\alpha_2)) + x\lambda_2^{2N}((f^{2N}-\alpha_1)\otimes 1\right]\hat{R}(x) =\\ &\quad = \left[x\lambda_2^{-2N}(1\otimes(f^{2N}-\alpha_1)) + \lambda_1^{2N}((f^{2N}-\alpha_2)\otimes 1\right]\hat{R}(x)\end{aligned} \tag{26}$$

on $M_{\lambda_1}\otimes M_{\lambda_2}$. So, $\hat{R}(x)(1\otimes(f^{2N}-\alpha_2))$ or $\hat{R}(x)((f^{2N}-\alpha_1)\otimes 1)$ can be represented as a linear combination of $(1\otimes(f^{2N}-\alpha_1))\hat{R}(x)$ and $((f^{2N}-\alpha_2)\otimes 1))\hat{R}(x)$ As f^{2N} is a central element of $U_q\widehat{osp}(1,2)$ in M_λ, we finished the proof.

Note that (25) for $x = 1$ coincides with the like one for $U_qsl(2,\mathbf{C})$, derived in [12, 13].

Obviously, the graded Yang-Baxter equations are satisfied on $V_{\lambda_1,\alpha_1}\otimes V_{\lambda_2,\alpha_2}\otimes V_{\lambda_3,\alpha_3}$ for the spectral parameter, lying on the algebraic curve

$$\frac{\alpha_1}{\lambda_1^{2N}-\lambda_1^{-2N}} = \frac{\alpha_2}{\lambda_2^{2N}-\lambda_2^{-2N}} = \frac{\alpha_3}{\lambda_3^{2N}-\lambda_3^{-2N}}, \qquad x^{2N} = 1, \qquad y^{2N} = 1$$

The spectral parameter of R-matrix lies on the same algebraic curve as in the case of $U_qsl(2,\mathbf{C})$ [12, 13].

In the case of $q^N = 1$ we can consider also the R-matrix (16) for the mixed tensor products $V_{\lambda,\alpha}\otimes V_j$ and $V_j\otimes V_{\lambda,\alpha}$ for $2j+1 < 2N$. Intertwiners of such type for $U_qsl(2,\mathbf{C})$ and $U_qsl(n,\mathbf{C})$ had been considered in [20, 21, 22]. It can be proven as above (see [17] for more detail for the case of $U_qsl(2,\mathbf{C})$) that the restriction of $\hat{R}(x)_{M_\lambda\otimes M_{q^{\frac{1}{2}j}}}$ $(\hat{R}(x)_{M_{q^{\frac{1}{2}j}}\otimes M_\lambda})$ to $V_{\lambda,\alpha}\otimes V_j$ $(V_j\otimes V_{\lambda,\alpha})$ is well defined for all values of parameters λ,α,x.

The graded Yang-Baxter equations are valid for mixed tenzor product of 3 representations because they are valid for corresponding Verma modules. If there are 2 semiperiodic representations V_{λ_1,α_1} and V_{λ_2,α_2}, then their parameters must lie on the algebraic curve (25)

Note that from any solution of graded Yang-Baxter equation the solution of ordinary (non-graded) one can be obtained [23]. The above considered R-matrices of $U_q osp(1,2)$ doesn't seem to give new solutions of Yang-Baxter equations as for spin-j irreps [18]. We suppose that they exhibit hidden symmetries in Chiral Potts and mixed models.

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